Today: finish up set theory.

The final 3 classes will be review | worksheet

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Last time: For two sets A, B, we say $|B| \le |A|$ if there is an injection $g: B \to A$.

Note: If g is injective, then $B \longrightarrow g(B)$ is a bijection.

image

Thm (Cantor-Schroeder-Bernstein):

If $|A| \le |B|$ and $|B| \le |A|$, then $|A| = |B|$

I.e. $\exists$ injection $f: A \to B$, $\exists$ injection $g: B \to A$ $\Rightarrow$ $\exists$ bijection $p: A \to B$

An equivalent statement:

take $C = g(B)$ and $h = g \circ f$

$$\boxed{\text{Thm: If } C \text{ is a subset of } A, \text{ and there is an injection } h: A \to C, \text{ then there is a bijection } p: A \to C}$$

Proof: Define the set $D \subseteq A$ by

$$D = (A-C) \cup h(A-C) \cup h^{\circ 2}(A-C) \cup \cdots$$
$$= \bigcup_{n=0}^{\infty} \underbrace{(h \circ \cdots \circ h)}_{n \text{ times}} (A-C).$$

Now define the function $p : A \to C$ by

$$\forall a \in A, \quad p(a) = \begin{cases} a & \text{if } a \notin D \\ h(a) & \text{if } a \in D. \end{cases}$$

Check that p is injective: ~~Essentially the~~

If $a \notin D$, then $p : A-D \longrightarrow A-D$ is the identity function ← injective always

If $a \in D$, then $p : D \longrightarrow D$ is the function h. ← injective by assumption.

Check that p is surjective:

Pick any $c \in C$. If $c \notin D$, then $c = h(c)$, so c is in the image of h.

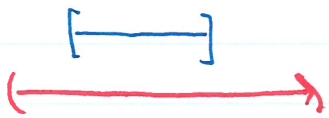
If $c \in D$, then $c \in \bigcup_{n=1}^{\infty} \underbrace{(h \circ \cdots \circ h)}_{n \text{ times}} (A-C)$ (because $c \notin A-C$)

Then $c \in \underbrace{(h \circ \cdots \circ h)}_{n \text{ times}} (A-C)$ for some $n \geq 1$. Then $c = h(a)$ for some $a \in \underbrace{(h \circ \cdots \circ h)}_{n-1 \text{ times}} (A-C)$

so c is in the image of h.

Example: The sets $[0,1]$ and $(0,1)$ have the same cardinality.

Proof:



Just consider the function
$$f : [0,1] \longrightarrow (0,1)$$
$$f(x) = \frac{x}{2} + \frac{1}{4}.$$

You can easily prove this is injective.
So by the CSB theorem, $|[0,1]| = |(0,1)|$.

Example: If $C \subseteq B \subseteq A$ and there's a bijection $f : A \to C$.

then $|A| = |B| = |C|$.

The size of $\mathbb{N}$ was called $\aleph_0$.

Def: The size of $\mathbb{R}$ is called c "continuum"

Question: Is there a set S such that $\underset{|\mathbb{N}|}{\underset{\shortparallel}{\aleph_0}} < |S| < \underset{|\mathbb{R}| = |P(\mathbb{N})|.}{\underset{\shortparallel}{c}}$?

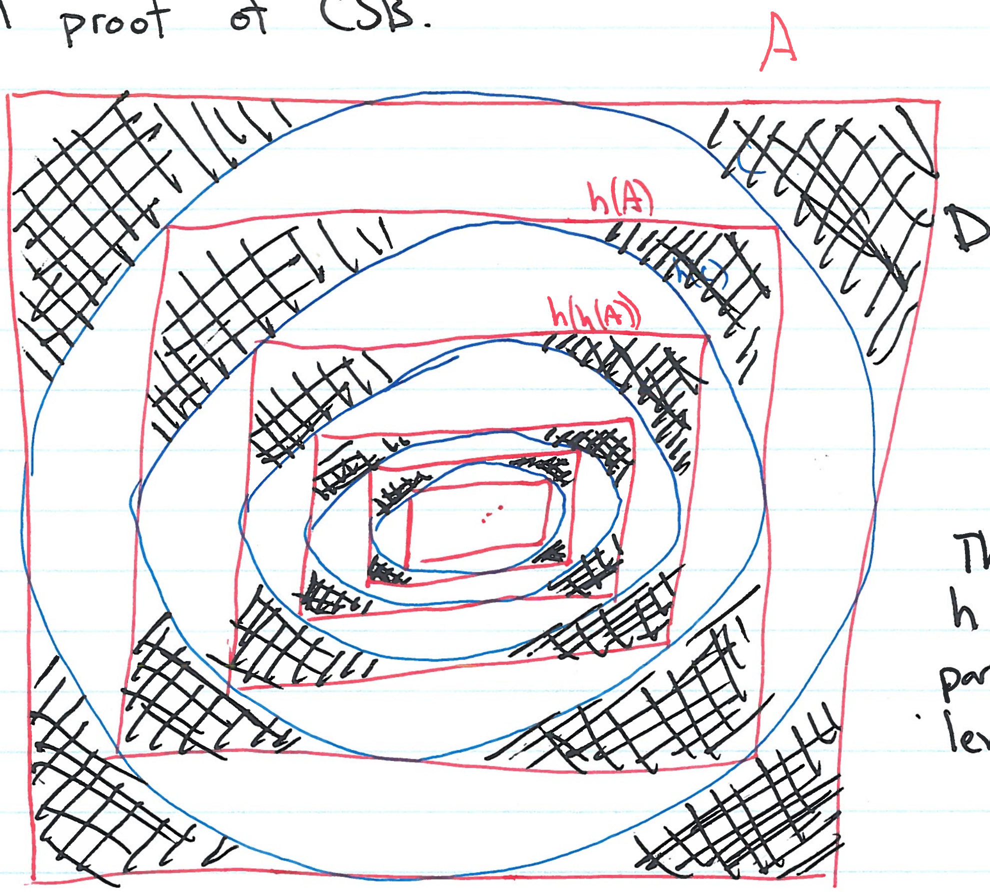
Many believe the answer is <u>no</u>.

(Continuum Hypothesis)

Godel (1940): The CH cannot be disproven using ZFC set theory.

Cohen (1963): The CH cannot be proven using ZFC set theory.

Pictorial proof of CSB.



The given function h shrinks every part inwards by one level.

Our new function p shrinks the # parts inwards by one level and leaves the other parts fixed.